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Solving the gauge identities

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Abstract. We show how one may 'solve' the Ward–Takahashi identities of a gauge theory to determine the longitudinal Green functions in terms of the basic source propagators and in such a way that on-shell amplitudes reduce to their classical values. We demonstrate the method for scalar, spinor and vector sources in Lorentz covariant and non-covariant Abelian gauges; but for non-Abelian groups we work in the axial gauge in order to avoid fictitious terms which otherwise spoil the procedure.

1. Introduction

Renormalisable quantum field theories which unify the basic forces of nature are founded on an underlying gauge principle and are consequently endowed with many attractive features, not the least being calculability. The gauge invariance possessed by the action finds its expression in the Ward–Takahashi identities (or their non-Abelian counterparts) connecting the various Green functions—relations between amplitudes involving an additional gauge vector current or extra gauge vector line. These gauge identities play such a crucial role in the renormalisation programme that one tries to preserve them at all costs by the regularisation scheme needed to define quantum loop corrections (although this is sometimes not possible for chiral groups) and, as a result, one can relate the various action counterterms and finish up with overall multiplicative renormalisations.

Such renormalisable gauge identities connect the divergence of an $(n+1)$-point amplitude with an n-point amplitude; therefore, for an amplitude with only two source legs and an arbitrary number of gauge lines, one can successively move down to the source propagator by taking a sufficient number of divergences at the vector ends. Working back, one can determine a good part of the Green functions (more precisely, the longitudinal pieces) in terms of the two-point propagator. This is the essence of the gauge technique (Salam 1963, Delbourgo and Salam 1964, Strathdee 1964) and it has the virtue of being a gauge-covariant procedure, suitable of being adaptable to any other approximation method for extracting solutions of the Green functions equations. Naturally there is vast ambiguity in the determination of the amplitudes since any transverse component (orthogonal to the contracting momentum) can be acceptably added without affecting the gauge covariance. As we shall see, these ambiguities can be effectively eliminated by requiring that on the source shell the amplitudes reduce to their classical values: this then provides the starting gauge approximation of the Green functions; the field equations provide subsequent transverse corrections.

In earlier papers the gauge technique was applied to scalar and spinor sources. We briefly recapitulate the method in § 2 and show how it can be generalised to vector particles (and higher-spin fields) in Lorentz covariant Abelian gauge theories; used in conjunction with Dyson–Schwinger equations we derive the infrared behaviour of charged spin-1 electrodynamics, paralleling the work on scalar and spinor electrodynamics (Delbourgo and West 1977a, b, Delbourgo 1977). The extension of the gauge technique to chiral groups and to (spontaneously broken) pseudovector electrodynamics is given in § 3.

Since unified gauge models centre round non-Abelian groups, the resulting Slavnov–Taylor identities in relativistic gauges (Slavnov 1972, Taylor 1971, Lee 1974) exhibit a rather complex form owing to the occurrence of fictitious particle terms; indeed, the latter render the gauge technique almost intractable since they include Bethe–Salpeter kernels of ghost-source scattering, whose spectral representations are barely known, and which can only be evaluated in a certain approximate sense†. Therefore for non-Abelian groups we prefer to stick to ghost-free axial gauges (Kummer 1961, Arnowitt and Feckler 1962, Schwinger 1963, Fradkin and Tyutin 1970) in which the Ward–Takahashi identities for the one-particle irreducible amplitudes do assume their characteristically simple form and become amenable to the gauge technique. One must, of course, pay the price of lack of relativistic invariance and we show how this can be met for electrodynamics in § 4 before going on to the Yang–Mills problem in § 5; the choice of axial gauge is vindicated by the simple structure of the full Green functions.

2. The gauge technique for electrodynamics

Let A_μ stand for the electromagnetic field, interacting with some quantised source field ϕ (and its adjoint $\bar{\phi}$), for which the connected vacuum generating functional W is defined through

$$\exp(\mathrm{i}W[j_\mu, j, \bar{j}]) = \int (\mathrm{d}A\,\mathrm{d}\phi\,\mathrm{d}\bar{\phi}) \exp\left(\mathrm{i}\int \mathrm{d}^4x(\mathscr{L}(A, \phi, \bar{\phi}) - j^\mu A_\mu - \bar{j}\phi - \bar{\phi}j - F(A))\right)$$

where $F(A)$ is some gauge fixing term—it needs no compensating in this Abelian case. The phase invariance of $\mathscr{L}$ under

$$\delta\phi = \mathrm{i}\Lambda\phi, \qquad \delta\bar{\phi} = -\mathrm{i}\bar{\phi}\Lambda, \qquad \delta A = -\partial\Lambda/e$$

results in the functional gauge identity,

$$\left[e\left(j\frac{\delta}{\delta j} - \bar{j}\frac{\delta}{\delta \bar{j}}\right) + \partial_\mu\left(\frac{\delta F}{\delta A_\mu}\right)\right]_{A_\nu = \mathrm{i}\delta/\delta j^\nu} W + \partial^\mu j = 0 \tag{1}$$

and functional current derivatives of the above equation yield the Ward–Takahashi identities between the connected Green functions

$$C(x_1, \ldots, x_n) \equiv \mathrm{i}^{n+1}\delta^n W/\delta j(x_1)\ldots\delta j(x_n).$$

For instance, in a covariant gauge specified by $F = -(\partial A)^2/2a$, one obtains the identity

$$a^{-1}\,\partial^\mu\,\partial^2 C_\mu(x; y, z) = e[\delta^4(x-z)C(y, x) - \delta^4(x-y)C(x, z)], \tag{2}$$

typical of the more general relation between $\partial C^{(n+1)}$ and $C^{(n)}$. One can arrive at

† Pagels (1976) has nevertheless done wonders with them by assuming $1/q^4$ behaviour of the gluon propagator and by picking out the infrared singular terms of the field equations.

similar identities for the one-particle irreducible Green functions $\Gamma^{(n)}$ defined through ($i\Delta \equiv C^{(2)} = \Gamma^{(2)-1}$),

$$C(x_1, \ldots, x_n) = \Big(\prod_j \int i\Delta(x_j, y_j)\, d^4y_j\Big)\, \Gamma(y_1, \ldots, y_n)$$

or the functional field derivatives of

$$\Gamma(A, \phi, \bar{\phi}) = W + \int (Aj + \bar{j}\phi + \bar{\phi}j).$$

For example,

$$\partial^\mu \Gamma_\mu(x; y, z) = e[\delta^4(x-z)\Gamma(y, x) - \delta^4(x-y)\Gamma(x, z)] \tag{3}$$

replaces identity (2), and so on. Extracting powers of e for each source line and taking Fourier transforms, we get the more familar versions of the identities† (Nishijima 1960, Rivers 1966),

$$\begin{aligned} &k^\mu \Gamma_\mu(p, p-k) = \Delta^{-1}(p) - \Delta^{-1}(p-k) \\ &k'^\nu \Gamma_{\nu\mu}(p'k', pk) = \Gamma_\mu(p+k, p) - \Gamma_\mu(p', p'-k) \end{aligned} \tag{4}$$

which were first derived via the canonical commutation formalism. These identities are multiplicately renormalised by the same (infinite) constant.

We aim to obtain 'solutions' of the gauge set (4) with the subsequent intention of inserting the solutions in the Green function equations. Gauge covariance is assured and not something which needs to be imposed at the end, making this the primary virtue of the gauge technique. Clearly an infinite number of possible 'solutions' can be found (Rivers 1966) all differing by transverse components that disappear upon contraction with the gauge field momentum; but this is not to deny that longitudinal components are interconnected by the identities and that equations (4) do embody considerable information about the Green functions. At this juncture it may be worth pointing out that in an axial gauge which specifies $n \,.\, A = 0$ (by the choice of gauge fixing term $F = Bn \,.\, A$, with B acting as a Lagrangian multiplier field), the identities (4) remain intact but identities like (2) are altered to orthogonality conditions

$$n^\mu C_\mu(x; y, z) = 0 \qquad \text{etc.}$$

These axial gauge identities (Delbourgo *et al* 1974, Kummer 1975) furthermore generalise very simply to non-Abelian groups, unlike the Lorentz covariant gauges where radical modifications become necessary.

One can arrive at a non-trivial class of solutions to (4) by noticing that: (i) the classical values of Γ (bare vertices) or of C (tree graphs) automatically obey the identities; (ii) successive divergences at the gauge legs bring us down to the two-point functions; and (iii) the propagators can themselves be represented as weighted spectral sums over free propagators. It follows that we can construct a gauge covariant set of quantum amplitudes by taking mass weighted sums of classical amplitudes. One could think of a more general procedure starting with a general representation of an N-point function and working up to the higher-point functions by some well defined algorithm—working down to lower-point functions is trivial—but this is extremely difficult to put into practice because spectral representations of fully off-shell amplitudes are hardly known or even guessed when $N > 3$. In any case it would be absurd

† Please note that there is no discrepancy between these identities and those quoted in previous researches. There the Γ correspond to amputated amplitudes, not one-particle irreducible parts.

to go to such lengths since the initial gauge approximation must be subject to transverse corrections entailed by the coupled Green functions equations, and therefore the starting point may as well be chosen simply. We believe we have done this by reverting in the end to the spectral form of the basic Born amplitudes with all gauge lines removed, namely the propagator, if there is a single source line. Besides, the renormalisations need only be carried out on the propagators and gauge-related vertex functions with the higher-point amplitudes then generated through the skeleton expansion.

In relativistically covariant gauges (axial gauges are considered below) where the propagators for scalar or spinor sources are rigorously known to possess the representations

$$\Delta(p)=\int \mathrm{d}W^2\rho(W)(p^2-W^2+\mathrm{i}0)^{-1}$$

or (5)

$$S(p)=\int \mathrm{d}W\rho(W)(\gamma\,.\,p-W+\mathrm{i}0\epsilon(W))^{-1},$$

the simplest gauge technique leads to the solutions

$$C(p',k_i,p)=\int \mathrm{d}W^2\rho(W^2)c(p',k_i,p|W^2)$$

or (6)

$$C(p',k_i,p)=\int \mathrm{d}W\rho(W)c(p',k_i,p|W),$$

where $c(\ldots|W)$ stand for the classical functions for a source of mass W. The first non-trivial illustration, the vertex function, explicitly reads

$$\Delta(p)\Gamma_\mu(p,p-k)\Delta(p-k)=\int \mathrm{d}W^2\rho(W^2)(p^2-W^2)^{-1}(2p-k)_\mu[(p-k)^2-W^2]$$

or (7)

$$S(p)\Gamma_\mu(p,p-k)S(p-k)=\int \mathrm{d}W\rho(W)[\gamma\,.\,p-W]^{-1}\gamma_\mu[\gamma\,.\,(p-k)-W]^{-1}$$

from which one readily sees how to write down the higher-point functions in this gauge approximation. One facet of this construction is that on-shell, where $\rho(W^2)\to\delta(W^2-m^2)$ or $\delta(W-m)$, the amplitudes become identically equal to the classical ones, and we may adopt this as the criterion which *defines* the initial gauge approximation. There only remains to find the spectral functions and this can be achieved via the Dyson–Schwinger equations as we have reported elsewhere; the procedure, which leads to an integral equation for ρ, is intrinsically non-perturbative. It also yields the infrared behaviour of the Green functions very economically (Delbourgo and West 1977b).

Electrodynamics of charged vector mesons is complicated by the occurrence of two spectral functions†

$$\Delta_{\mu\nu}(p)=\int \mathrm{d}W^2[(-\eta_{\mu\nu}+p_\mu p_\nu/W^2)\rho(W^2)-\eta_{\mu\nu}\tau(W^2)](p^2-W^2+\mathrm{i}0)^{-1} \qquad (8)$$

† Actually the spinor case implicitly contains two spectral functions, the even and odd parts of $\rho(W)$, and it suggests that a closer analogy with mesons can be achieved by using spectral representations in Kemmer's β-formalism rather than (8).

but the solution here is readily understood if one or other of the weights is taken to be zero in turn. Thus $\tau = 0$ corresponds to an integral over spin-1 vector mesons of mass W whose field theory is governed by

$$\mathscr{L}_\rho = -\tfrac{1}{2}(D_\mu V_\nu^+ - D_\nu V_\mu^+)(D^\mu V^\nu - D^\nu V^\mu) + W^2 V_\mu^+ V^\mu$$

with bare vertices

$$\begin{aligned} R_{\lambda\mu\nu}(p, p-k) &= -\eta_{\mu\nu}(2p-k)_\lambda + (p-k)_\mu \eta_{\nu\lambda} + p_\nu \eta_{\mu\lambda}, \\ R_{\kappa\lambda\mu\nu} &= -2\eta_{\mu\nu}\eta_{\kappa\lambda} + \eta_{\mu\kappa}\eta_{\nu\lambda} + \eta_{\nu\kappa}\eta_{\mu\lambda}, \end{aligned} \tag{9}$$

whereas $\rho = 0$ refers to an (unphysical) theory

$$\mathscr{L}_\tau = -D_\mu V_\nu^+ D^\mu V^\nu + W^2 V_\mu^+ V^\mu$$

having for its bare $\Gamma^{(n)}$,

$$T_{\lambda\mu\nu}(p, p-k) = -\eta_{\mu\nu}(2p-k)_\lambda, \qquad T_{\kappa\lambda\mu\nu} = -2\eta_{\mu\nu}\eta_{\kappa\lambda}. \tag{10}$$

After these observations it comes as no surprise that the gauge identities are solved by

$$\begin{aligned} &\Delta^{\mu\mu'}(p)\Gamma_{\lambda\mu'\nu'}(p, p-k)\Delta^{\nu'\nu}(p-k) \\ &= \int \frac{\mathrm{d}W^2}{(p^2 - W^2)[(p-k)^2 - W^2]} \left[\left(\eta^{\mu\mu'} - \frac{p^\mu p^{\mu'}}{W^2}\right) R_{\lambda\mu'\nu'} \right. \\ &\quad \left. \times \left(\eta^{\nu'\nu} - \frac{(p-k)^{\nu'}(p-k)^\nu}{W^2}\right) \rho(W^2) - T_{\lambda\mu\nu}\tau(W^2)\right] \end{aligned} \tag{11}$$

as one can easily check; and similarly for the higher-point functions. In the appendix we have pursued (11) and determined the integral equations for ρ and τ that are provided by the field equations. In the infrared limit we find $\tau \to 0$ and $\rho \to (W^2 - m^2)^{-1-e^2(3-a)/8\pi^2}$ in complete analogy to the scalar and spinor cases, which strongly suggests spin independence of infrared behaviour.

3. Pseudovector electrodynamics

Next we consider abnormal (1^{++}) photons. For simplicity and also for aesthetic reasons we suppose that the chiral symmetry is exact at the Lagrangian level so that its eventual breaking is spontaneous or dynamical but not as the result of quantum regularisation, i.e. we introduce enough sources to cancel out offending anomalies. Being a true symmetry of the action the chiral U(1) group leads to its own set of gauge identities, quite analogous to (1). For fermion pseudovector electrodynamics where

$$\mathscr{L}_5 = \bar{\psi}\mathrm{i}\gamma(\partial + eA\gamma_5)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

the generalisations of (4) read

$$\begin{aligned} -\mathrm{i}k^\mu \Gamma_{\mu 5}(p \,.\, p-k) &= S^{-1}(p)\gamma_5 + \gamma_5 S^{-1}(p-k) \\ \mathrm{i}k^\mu \Gamma_{\nu 5 \mu 5}(p'k', pk) &= \Gamma_{\nu 5}(p', p'+k')\gamma_5 + \gamma_5 \Gamma_{\nu 5}(p-k', p) \end{aligned} \tag{12}$$

etc, and are trivially obeyed at the bare level (massless fermions, $\Gamma_{\mu 5} = \mathrm{i}\gamma_\mu\gamma_5$, $\Gamma^{(n)} = 0$ for $n > 3$). After quantum corrections however, $S(p)$ propagates with a whole spectrum of intermediate massive fermions from which it necessarily follows that $\Gamma_{\mu 5}$

contains a pole; this implies dynamical chiral breakdown (Jackiw and Johnson 1973) and the generation of a pseudovector photon mass through Schwinger's mechanism (1962). Thus

$$\gamma_5 S(p-k)+S(p)\gamma_5$$
$$=\int \mathrm{d}W\rho(W)\{\gamma_5[\gamma\,.\,(p-k)-W]^{-1}+(\gamma\,.\,p-W)^{-1}\gamma_5\}$$
$$=\int \mathrm{d}W\rho(W)(\gamma\,.\,p-W)^{-1}(k\,.\,\gamma-2W)\gamma_5[\gamma\,.\,(p-k)-W]^{-1}$$

provides the solution,

$$S(p)\Gamma_{\mu5}(p,p-k)S(p-k)=\int \mathrm{d}W\rho(W)\frac{1}{\gamma\,.\,p-W}\left(\mathrm{i}\gamma_\mu\gamma_5-\frac{2\mathrm{i}Wk_\mu\gamma_5}{k^2}\right)\frac{1}{\gamma\,.\,(p-k)-W} \tag{13}$$

manifesting the $k^2\to 0$ singularity with all its consequences.

The solutions of the gauge identities for the higher-point amplitudes can be extracted from the work of Jackiw and Johnson (1973). Those authors noted that the phenomenological Lagrangian

$$\mathscr{L}_{5M}=\bar{\psi}\gamma\,.\,(\partial+eA)\gamma_5\psi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}\mu^2A^2-\mu A\,.\,\partial\phi+\tfrac{1}{2}(\partial\phi)^2-m\bar{\psi}\exp(2e\gamma_5\phi/\mu)\psi$$

of *massive* mesons and fermions also possesses a local chiral symmetry, it being the coupling of A to the massless ϕ which is responsible for the $1/k^2$ poles at the meson legs. By drawing all the tree graphs of this theory and suitably summing over masses, one arrives at the requisite solution. For instance, the four-point amplitude, with vector lines amputated, reads

$$G_{\nu5\mu5}(p'k',pk)$$
$$=\int \mathrm{d}W\rho(W)\frac{1}{\gamma\,.\,p'-W}\left[\left(\gamma_\nu\gamma_5-\frac{2Wk'_\nu\gamma_5}{k'^2}\right)\frac{1}{\gamma\,.\,(p+k)-W}\right.$$
$$\times\left(\gamma_\mu\gamma_5-\frac{2Wk_\mu\gamma_5}{k^2}\right)+\left(\gamma_\mu\gamma_5-\frac{2Wk_\mu\gamma_5}{k^2}\right)\frac{1}{\gamma\,.\,(p'-k)-W}\left(\gamma_\nu\gamma_5-\frac{2Wk'_\nu\gamma_5}{k'^2}\right)$$
$$\left.+\frac{4Wk_\mu k'_\nu}{k^2k'^2}\right]\frac{1}{\gamma\,.\,p-W}\,.$$

This initial gauge approximation can be taken as a suitable basis for a self-consistent determination of $\rho(W)$, but the evaluation is considerably harder than normal vector electrodynamics because the self-energy corrections of the pseudovector lines *cannot* be dropped in the first instance as they serve to render the meson massive and thereby alter the entire cut structure of $S(p)$. Parallel comments apply to scalar and vector sources.

4. Electrodynamics in axial gauges

We shall stop using Lorentz covariant gauges from now on. The reason is that we are placing our entire emphasis on the gauge identities, so it is crucial for us that they be as simple as possible. Non-Abelian groups in relativistic invariant groups are known to lead to highly complicated Slavnov–Taylor identities riddled with fictitious particle terms and various Bethe–Salpeter kernels; these make our task of solving the identities well nigh impossible and force us into adopting an axial gauge where the Green functions cease to be Lorentz covariant (they depend on an external four-vector n) but where the identities for $\Gamma^{(n)}$ are straightforward (Bernstein 1977, Delbourgo *et al* 1974, Kummer 1975). As an introduction to Yang–Mills theory let us first study electrodynamics.

As we mentioned, in the gauge $n \, . \, A=0$, the Ward–Takahashi identities for one-particle irreducible amplitudes are the naive ones obtained by canonical methods, i.e. the set (4). The one and only problem is that the $\Gamma^{(n)}$ are functions of n as well as the momenta; e.g. in the scalar case, $\Delta(p, n)=\mathcal{O}(p^2, p \, . \, n)$. The initial gauge approximation writes the $\Gamma^{(n)}$ as functionals of Δ, and it is therefore essential to know something about the cut structure of the propagators in axial gauges. An examination of the source self-energy past $\Pi(p, n)$ in lowest-order perturbation theory reveals quite a lot: Π is a function of p^2 and $(p \, . \, n)^2$ and when $(p \, . \, n)^2 < m^2$, there is just the usual relativistic cut for $p^2 \geqslant m^2$. It is a strong indication that the correct representation† of the propagator is

$$\Delta(p, n)=\int \mathrm{d}W^2 \rho(W^2, p \, . \, n)(p^2-W^2+\mathrm{i}0)^{-1} \tag{15}$$

providing $(p \, . \, n)^2 < W^2$ (threshold). Note how the covariant gauge parameter is replaced by the parameter $p \, . \, n/m$ in the axial gauge. With the fermion self-energy, one finds instead

$$\Sigma(p, n)=\gamma \, . \, p\Sigma_1(p^2, p \, . \, n)+m\Sigma_2(p^2, p \, . \, n)+\gamma \, . \, n\Sigma_0(p^2, p \, . \, n)$$

which in turn leads to the spectral form

$$S(p, n)=\tfrac{1}{2}\int \mathrm{d}W[(\gamma \, . \, p-W)^{-1}, \rho(W, p \, . \, n)+\gamma \, . \, n\rho_0(W, p \, . \, n)] \tag{16}$$

if $(p \, . \, n)^2 < m^2$. The absence of terms $[\gamma \, . \, n, \gamma \, . \, p]$ in S or Σ is a consequence of C-invariance, since the Lagrangian multiplier field B like A_μ possesses negative charge parity. We defer the vector case to the next section.

Finding 'solutions' of the Ward–Takahashi identities is complicated by the $p \, . \, n$ dependence of the spectral functions. For scalar electrodynamics we can tackle the difficulty by writing the propagator difference as

$$\begin{aligned}
\Delta(p)-\Delta(p') &= \int \mathrm{d}W^2 \left(\frac{\rho(W^2, p \, . \, n)}{p^2-W^2}-\frac{\rho(W^2, p' \, . \, n)}{p'^2-W^2}\right) \\
&= \tfrac{1}{2}\int \mathrm{d}W^2 \left[\left(\frac{1}{p^2-W^2}-\frac{1}{p'^2-W^2}\right)(\rho(W^2, p \, . \, n)+\rho(W^2, p' \, . \, n)) \right. \\
&\quad \left. +\left(\frac{1}{p^2-W^2}+\frac{1}{p'^2-W^2}\right)(\rho(W^2, p \, . \, n)-\rho(W^2, p' \, . \, n))\right].
\end{aligned}$$

† To our knowledge a *rigorous* proof of (15) or (16) is lacking, but this has not prevented its use (Frenkel and Taylor 1976).

Then, by inspection, an appropriate solution of the vertex identity,

$$\Delta(p)\Gamma_\mu(p,p')\Delta(p')$$

$$=\tfrac{1}{2}\int \mathrm{d}W^2\,\frac{1}{p^2-W^2}(p+p')_\mu\frac{1}{p'^2-W^2}(\rho(W^2,p\,.\,n)+\rho(W^2,p'\,.\,n))$$

$$-\tfrac{1}{2}\int \mathrm{d}W^2\left(\frac{1}{p^2-W^2}+\frac{1}{p'^2-W^2}\right)n_\mu\,\frac{\rho(W^2,p\,.\,n)-\rho(W^2,p'\,.\,n)}{p\,.\,n-p'\,.\,n} \tag{17}$$

is rightly Bose symmetric. Observe also that the n_μ component disappears from the full Green function (multiplication by $D^{\lambda\mu}(p-p')$) so that the only true signal of non-covariance is the n-dependence of the averaged spectral function. The same analysis can be carried over to the higher-point functions and again it is only the mean ρ which enters the expressions.

With spinors the decomposition problem is not more difficult. Write

$$S(p)-S(p')=\tfrac{1}{4}\int \mathrm{d}W\left\{\frac{1}{\gamma\,.\,p-W}-\frac{1}{\gamma\,.\,p'-W},\rho+\gamma\,.\,n\rho_0+\rho'+\gamma\,.\,n\rho_0'\right\}$$

$$+\tfrac{1}{4}\int \mathrm{d}W\left\{\frac{1}{\gamma\,.\,p-W}+\frac{1}{\gamma\,.\,p'-W},\rho+\gamma\,.\,n\rho_0-\rho'-\gamma\,.\,n\rho_0'\right\}.$$

Then again,

$$S(p)\Gamma_\mu(p,p')S(p')$$

$$=\tfrac{1}{4}\int \mathrm{d}W\left\{\frac{1}{\gamma\,.\,p-W}\gamma_\mu\frac{1}{\gamma\,.\,p'-W},\rho+\gamma\,.\,n\rho_0+\rho'+\gamma\,.\,n\rho_0'\right\}$$

$$-\tfrac{1}{4}\int \mathrm{d}W\,n_\mu\left\{\frac{1}{\gamma\,.\,p-W}+\frac{1}{\gamma\,.\,p'-W},\frac{\rho+\gamma\,.\,n\rho_0-\rho-\gamma\,.\,n\rho_0'}{p\,.\,n-p'\,.\,n}\right\} \tag{18}$$

and so on to higher-point functions, with n_μ pieces being essentially irrelevant. These gauge covariant expressions are ready for redeployment in the field equations, but that exercise lies outside the scope of this paper. Rather, we pass on to discovering solutions for the more interesting non-Abelian problem.

5. Non-Abelian theories in axial gauges

Let A_μ stand for the gauge meson fields in the regular representation of the internal symmetry and let f^{abc} be the structure constants. In axial gauges $n\,.\,A^a=0$, the connecting Green functions involving at least two vector lines,

$$C^{a_1a_2\cdots}_{\mu_1\mu_2\cdots}(x_1,x_2,\ldots)=\delta^{\cdots}W/\delta j^{a_1}_{\mu_1}(x_1)\delta j^{a_2}_{\mu_2}(x_2)\ldots,$$

including any number of source derivatives $\delta/\delta j$, $\delta/\delta K$ (where K is the current of the multiplier field B), obey

$$n^{\mu_i}C_{\cdot\mu_i\cdot}(\,\cdot\,,x_i,\,\cdot\,)=0 \tag{19}$$

because of f-antisymmetry†. A particular case is the propagator

$$\mathrm{i}\delta^{ab}\Delta_{\mu\nu}(x-y)=\delta^2W/\delta j^a_\mu(x)\delta j^b_\nu(y);\qquad n^\mu\Delta_{\mu\nu}=0,$$

† And the basic functional gauge equation $(D_\mu j^\mu+\mathrm{i}n\,.\,D\,\delta/\delta K)W=0$.

which together with the mixed propagator

$$\mathrm{i}\delta^{ab}\Delta_{\mu B}(x-y)=\delta^2 W/\delta j^a_\mu(x)\delta K^b(y)=\mathrm{i}\delta^{ab}(n\,.\,\partial)^{-1}\,\partial_\mu\delta^4(x-y) \qquad (20)$$

and

$$\mathrm{i}\Delta_{BB}(x-y)=\delta^2 W/\delta K^a(x)\delta K^b(y)=0$$

make up the full A_μ, B propagator matrix. The result (19) is not very surprising in view of the fact that $C_{\cdot\mu\cdot}(\cdot\ x\ \cdot)$ factorises as $\int\Delta_{\mu\nu}(x-y)\Gamma_{\cdot\nu\cdot}(\cdot\ y\ \cdot)\,\mathrm{d}^4y$.

The $\Gamma^{(n)}$ functions in the pure gauge theory satisfy the canonical identities

$$\begin{aligned}
&\partial^\lambda(\Delta^{-1})^{ab}_{\lambda\mu}(x-y)=0\\
&\partial^\lambda\Gamma^{abc}_{\lambda\mu\nu}(xyz)=f^{abe}\delta^4(x-y)(\Delta^{-1})^{ce}_{\mu\nu}(x-z)+f^{ace}\delta^4(x-z)(\Delta^{-1})^{be}_{\mu\nu}(y-x)\\
&\partial^\kappa\Gamma^{abcd}_{\kappa\lambda\mu\nu}(xyzw)=f^{abe}\delta^4(x-y)\Gamma^{ecd}_{\lambda\mu\nu}(xzw)+f^{ace}\delta^4(x-z)\Gamma^{bed}_{\lambda\mu\nu}(yxw)\\
&\qquad\qquad+f^{ade}\delta^4(x-w)\Gamma^{bce}_{\lambda\mu\nu}(yxw)
\end{aligned} \qquad (21)$$

etc. And even if additional sources ϕ_i—on which the generators are represented by matrices $(T^a)^j_i$—are incorporated, the identities retain their traditionally simple form,

$$\partial^\lambda\Gamma^a_\lambda(x;\,y,\,z)=\delta^4(x-y)\Delta^{-1}(x-z)T^a-T^a\delta^4(x-z)\Delta^{-1}(y-x). \qquad (22)$$

How do we go about solving such involved relations as (21)? Many clues have been offered in previous sections, but before they can be applied one should appreciate that the multiplier field plays an important role and that the full propagator matrix Δ and its inverse Δ^{-1}, with elements $(\Delta^{-1})^{ab}_{\mu\nu}$, $\Delta^{-1}({}^{ab}_{B\mu})=n_\mu\delta^{ab}$ (remaining unrenormalised like $\Delta_{B\mu}$) and $(\Delta^{-1})_{BB}=0$, figure in many places. Also recall that $\Gamma_{B\mu_1\mu_2\ldots}=0$ for one-particle irreducible amplitudes comprising at least two vectors. Let us therefore start with the spectral representation†, expected to be valid when $(p\,.\,n)^2<0$,

$$\begin{aligned}
\Delta_{\mu\nu}(p)=\int \mathrm{d}W^2(p^2-W^2)^{-1}\bigg[&\left(-\eta_{\mu\nu}+\frac{p_\mu n_\nu+n_\mu p_\nu}{p\,.\,n}-\frac{p_\mu p_\nu}{(p\,.\,n)^2}\right)\alpha(W^2,p\,.\,n)\\
&+\left(-\eta_{\mu\nu}+\frac{n_\mu n_\nu}{n^2}\right)\beta(W^2,p\,.\,n)\bigg]
\end{aligned} \qquad (23)$$

and of course the unaffected parts,

$$\Delta_{B\mu}=p_\mu/(p\,.\,n), \qquad \Delta_{BB}=0. \qquad (24)$$

The bare propagator is obtained by substituting $\alpha(W^2)\to\delta(W^2)$, $\beta(W^2)\to 0$. Our objective is to write down all higher-point functions in terms of α and β so as to satisfy identities (21), and to achieve it we will have to synthesise the methods of §§ 2 and 4 in a manner which represents the *full Bose symmetry* of the amplitudes.

For the moment, forget about the n-dependence of α and β. When $\beta=0$, one is dealing with a massive vector theory with inverse propagator

$$\Delta^{-1}_{\mu\nu}(p)=(p^2-W^2)(-\eta_{\mu\nu}+p_\mu p_\nu/p^2)\equiv\mathscr{A}^{-1}_{\mu\nu}, \qquad \Delta^{-1}_{B\mu}(p)=n_\mu, \qquad \Delta^{-1}_{BB}=0. \qquad (25)$$

Since

$$\Delta^{-1}_{\mu\nu}(q)-\Delta^{-1}_{\mu\nu}(r)=p^\lambda\Lambda^0_{\lambda\mu\nu}(p,q,r)+W^2(r_\mu r_\nu/r^2-q_\mu q_\nu/q^2) \qquad (26)$$

† We have extracted the internal factor δ^{ab} out of Δ^{ab}; likewise below we remove the factor f^{abc} from Γ^{abc}, always assuming the gauge symmetry is not spontaneously broken.

where

$$\Lambda^0_{\lambda\mu\nu}(p,q,r) = -\eta_{\mu\nu}(q-r)_\lambda - \eta_{\mu\nu}(r-p)_\mu - \eta_{\lambda\mu}(p-q)_\nu \tag{27}$$

is the bare Yang–Mills vertex, an acceptable solution is

$$\Delta^{\mu\mu'}(q)\Gamma_{\lambda\mu'\nu'}(p,q,r)\Delta^{\nu'\nu}(r)$$

$$= \int \mathrm{d}W^2 \alpha(W^2)\left(-\eta^{\mu\mu'} + \frac{q^{\mu'}n^\mu + q^\mu n^{\mu'}}{q\,.\,n} - \frac{q^\mu q^{\mu'}}{(q\,.\,n)^2}\right)\frac{1}{q^2 - W^2}$$

$$\times \Lambda^{(a)}_{\lambda\mu'\nu'}\left(-\eta^{\nu'\nu} + \frac{r^{\nu'}n^\nu + r^\nu n^{\nu'}}{r\,.\,n} - \frac{r^{\nu'}r^\nu}{(r\,.\,n)^2}\right)\frac{1}{r^2 - W^2}$$

with

$$\Lambda^{(a)}_{\lambda\mu\nu} = \Lambda^0_{\lambda\mu\nu} + W^2\left[\frac{n_\lambda}{p\,.\,n}\left(\frac{r_\mu r_\nu}{r^2} - \frac{q_\mu q_\nu}{q^2}\right) + (p\lambda \leftrightarrow q\mu \leftrightarrow r\nu \text{ perms})\right]. \tag{28}$$

A similar analysis, when $\alpha = 0$ but $\beta \neq 0$, gives

$$\Delta^{\mu\mu'}(q)\Gamma_{\lambda\mu'\nu'}(pqr)\Delta^{\nu'\nu}(r)$$

$$= \int \mathrm{d}W^2 \beta(W^2)\frac{-\eta^{\mu\mu'} + n^\mu n^{\mu'}/n^2}{q^2 - W^2}\,\Lambda^{(b)}_{\lambda\mu'\nu'}\,\frac{-\eta^{\nu'\nu} + n^{\nu'}n^\nu/n^2}{r^2 - W^2}$$

with

$$\Lambda^{(b)}_{\lambda\mu\nu} = \Lambda^0_{\lambda\mu\nu} + n_\lambda(r_\mu r_\nu - q_\mu q_\nu)/p\,.\,n + \text{perms}. \tag{29}$$

The lack of total Bose symmetry in $\Lambda^{(a)}$ and $\Lambda^{(b)}$ has been remedied by adding supplementary terms, proportional to n_μ and n_ν, which vanish upon contraction with $\Delta(q)$ and $\Delta(r)$. Observe too that the n_λ-pieces themselves will disappear when $\Delta(p)$ is applied on (28) and (29) to obtain the full Green function, but that they cannot be dropped if we require the mixed function $\Delta^{B\lambda}(p)\Gamma_{\lambda\mu\nu}$.

A more serious obstacle is presented by the $p\,.\,n$ dependence of the spectral functions. To circumvent this, define the components,

$$\begin{aligned}\mathscr{A}^{\mu\nu}(p, W) &\equiv \left(-\eta^{\mu\nu} + \frac{p^\mu n^\nu + p^\mu n^\nu}{p\,.\,n} - \frac{p^\mu p^\nu}{(p\,.\,n)^2}\right)\frac{1}{p^2 - W^2} \\ \mathscr{B}^{\mu\nu}(p, W) &\equiv (-\eta^{\mu\nu} + n^\mu n^\nu/n^2)/(p^2 - W^2)\end{aligned} \tag{30}$$

and set $\beta = 0$ at first. Then

$$[\Delta(q)p\,.\,\Gamma\Delta(r)]_{\mu\nu}$$

$$= \tfrac{1}{2}\int \mathrm{d}W^2[\alpha(W^2, q\,.\,n) + \alpha(W^2, r\,.\,n)][\mathscr{A}(q, W)p\,.\,\Lambda^a\mathscr{A}(r, W)]_{\mu\nu}$$

$$+ \tfrac{1}{2}\int \mathrm{d}W^2[\alpha(W^2, r\,.\,n) - \alpha(W^2, q\,.\,n)][\mathscr{A}(q, W) + \mathscr{A}(r, W)]_{\mu\nu} \tag{31}$$

immediately suggests a possible factorisation of p, following the manoeuvre of § 4. Unfortunately (31) is not yet ready for the operation since the straight recipe would not provide a totally symmetric Γ. Instead we must carry on with our manipulations on (31) and write

$$[\Delta(q)p\,.\,\Gamma\Delta(r)]_{\mu\nu}$$

$$=\tfrac{1}{3}\int \mathrm{d}W^2[\alpha(W^2,q\,.\,n)+\alpha(W^2,r\,.\,n)$$

$$+\alpha(W^2,p\,.\,n)][\mathscr{A}(q,W)p\,.\,\Lambda^a\mathscr{A}(r,W)]_{\mu\nu}$$

$$+\tfrac{1}{6}\int \mathrm{d}W^2[\alpha(W^2,q\,.\,n)+\alpha(W^2,r\,.\,n)$$

$$-2\alpha(W^2,p\,.\,n)][\mathscr{A}(r,W)-\mathscr{A}(q,W)]_{\mu\nu}$$

$$+\tfrac{1}{2}\int \mathrm{d}W^2[\alpha(W^2,r\,.\,n)-\alpha(W^2,q\,.\,n)][\mathscr{A}(r,W)+\mathscr{A}(q,W)]_{\mu\nu}.$$

We deduce that

$$[\Delta(q)\Gamma\Delta(r)]_{\lambda\mu\nu}$$

$$=\tfrac{1}{3}\int \mathrm{d}W^2[\alpha(W^2,q\,.\,n)+\alpha(W^2,r\,.\,n)$$

$$+\alpha(W^2,p\,.\,n)][\mathscr{A}(q,W)\Lambda^{(a)}\mathscr{A}(r,W)]_{\lambda\mu\nu}$$

$$+\frac{n_\lambda}{6p\,.\,n}\int \mathrm{d}W^2[\alpha(W^2,q\,.\,n)+\alpha(W^2,r\,.\,n)$$

$$-2\alpha(W^2,p\,.\,n)][\mathscr{A}(r,W)-\mathscr{A}(q,W)]_{\mu\nu}$$

$$+\frac{n_\lambda}{2p\,.\,n}\int \mathrm{d}W^2[\alpha(W^2,r\,.\,n)-\alpha(W^2,q\,.\,n)][\mathscr{A}(r,W)+\mathscr{A}(q,W)]_{\mu\nu}$$

up to n_μ, n_ν terms in Γ.

Having outlined the essential steps we may reasonably quote the complete, symmetrised answer:

$$\Delta^{\lambda\lambda'}(p)\Delta^{\mu\mu'}(q)\Delta^{\nu\nu'}(r)\Gamma_{\lambda'\mu'\nu'}(pqr)$$

$$=\int \mathrm{d}W^2\mathscr{A}^{\lambda\lambda'}(p,W)\mathscr{A}^{\mu\mu'}(q,W)\mathscr{A}^{\nu\nu'}(r,W)\Lambda^{\alpha}_{\lambda'\mu'\nu'}(pqr|Wn)$$

$$+\int \mathrm{d}W^2\mathscr{B}^{\lambda\lambda'}(p,W)\mathscr{B}^{\mu\mu'}(q,W)\mathscr{B}^{\nu\nu'}(r,W)\Lambda^{\beta}_{\lambda'\mu'\nu'}(pqr|Wn) \qquad (32)$$

in which $\mathscr{A}$ and $\mathscr{B}$ are given by (30) and

$$\Lambda^{\alpha}_{\lambda\mu\nu}(pqr|Wn)=\tfrac{1}{3}[\alpha(W^2,p\,.\,n)+\alpha(W^2,q\,.\,n)+\alpha(W^2,r\,.\,n)]\Lambda^{(a)}_{\lambda\mu\nu}$$

$$+\frac{n_\lambda}{6p\,.\,n}[\alpha(W^2,q\,.\,n)+\alpha(W^2,r\,.\,n)$$

$$-2\alpha(W^2,p\,.\,n)][\mathscr{A}^{-1}(q,W)-\mathscr{A}^{-1}(r,W)]_{\mu\nu}$$

$$+\frac{n_\lambda}{2p\,.\,n}[\alpha(W^2,r\,.\,n)-\alpha(W^2,q\,.\,n)][\mathscr{A}^{-1}(q,W)+\mathscr{A}^{-1}(r,W)]_{\mu\nu}$$

$$+(p\lambda\leftrightarrow q\mu\leftrightarrow r\nu\ \text{perms}) \qquad (33)$$

$$\Lambda^{\beta}_{\lambda\mu\nu}(pqr|Wn)=\tfrac{1}{3}[\beta(W^2,p.n)+\beta(W^2,q.n)+\beta(W^2,r.n)]\Lambda^{(b)}_{\lambda\mu\nu}$$
$$+\frac{n_\lambda}{6p.n}[\beta(W^2,q.n)+\beta(W^2,r.n)$$
$$-2\beta(W^2,p.n)][\mathscr{B}^{-1}(r,W)-\mathscr{B}^{-1}(q,W)]_{\mu\nu}$$
$$+\frac{n_\lambda}{2p.n}[\beta(W^2,r.n)-\beta(W^2,q.n)][\mathscr{B}^{-1}(r,W)+\mathscr{B}^{-1}(q,W)]_{\mu\nu}$$
$$+(p\lambda\leftrightarrow q\mu\leftrightarrow r\nu\text{ perms}). \tag{34}$$

As (32) stands, we can discard all n-pieces from Λ^{α} and Λ^{β} to leave the *relativistically covariant Yang–Mills vertex* Λ^0 weighted by the *average spectral function*,

$$[\Delta(p)\Delta(q)\Delta(r)\Gamma(pqr)]_{\lambda\mu\nu}$$
$$=\int dW^2\tfrac{1}{3}[\alpha(W^2,p.n)+\alpha(W^2,q.n)+\alpha(W^2,r.n)][\mathscr{A}(p)\mathscr{A}(q)\mathscr{A}(r)\Lambda^0]_{\lambda\mu\nu}$$
$$+\int dW^2\tfrac{1}{3}[\beta(W^2,p.n)+\beta(W^2,q.n)$$
$$+\beta(W^2,r.n)][\mathscr{B}(p)\mathscr{B}(q)\mathscr{B}(r)\Lambda^0]_{\lambda\mu\nu} \tag{35}$$

a pleasingly simple answer for the only Green function which really matters. The n-terms in Λ^{α} and Λ^{β} come into their own for the other Green functions which are intimately tied to the gauge identities:

$$\Delta^{B\lambda}(p)\Delta^{\mu\mu'}(p)\Delta^{\nu\nu'}(r)\Gamma_{\lambda\mu'\nu'}(pqr)$$
$$=(p.n)^{-1}[\Delta(q)p.\Gamma\Delta(r)]^{\mu\nu}$$
$$=(p.n)^{-1}[\Delta^{\mu\nu}(r)-\Delta^{\mu\nu}(q)]$$
$$=\int dW^2(p.n)^{-1}p^{\lambda}\mathscr{A}^{\mu\mu'}(q,W)\mathscr{A}^{\nu\nu'}(r,W)\Lambda^{\alpha}_{\lambda\mu'\nu'}(pqr|Wn)$$
$$+\int dW^2(p.n)^{-1}p^{\lambda}\mathscr{B}^{\mu\mu'}(q,W)\mathscr{B}^{\nu\nu'}(r,W)\Lambda^{\beta}_{\lambda\mu'\nu'}(pqr|Wn).$$

The solution (35) is tailor-made for insertion in the Dyson–Schwinger equations and a proper treatment of the (non-perturbative) infrared behaviour of the gluon propagator. However, a first investigation of the Abelian problem will be needed to set the scene.

Appendix. Infrared behaviour of charged vector meson theory

The Dyson–Schwinger equations in vector electrodynamics include the source equation

$$Z^{-1}\delta^{\nu}_{\mu}=\Delta_{\mu\lambda}(p)[-(p^2-m_0^2)\eta^{\lambda\nu}+p^{\lambda}p^{\nu}]$$
$$-ie^2\int \bar{d}^4k D_{\lambda\lambda'}(k)\Delta_{\mu\mu'}(p)\Gamma^{\lambda'\mu'\nu'}(p,p-k)\Delta_{\nu'\rho}(p-k)R^{\lambda\nu\rho}(p,p-k)$$
$$+\text{tadpole term}+\text{two-photon contribution} \tag{A.1}$$

wherein

$$R_{\lambda\mu\nu} = -\eta_{\mu\nu}(2p-k)_\lambda + (p-k)_\mu\eta_{\nu\lambda} + p_\nu\eta_{\mu\lambda} - M(k_\mu\eta_{\nu\lambda} - k_\nu\eta_{\mu\lambda}) \quad \text{(A.2)}$$

is the bare vertex (9), obtained by minimal substitution, to which we have added a magnetic moment M. The luxury of magnetic interactions must be afforded if vector electrodynamics is eventually regarded as part of a Yang–Mills system, when $M=1$ becomes the norm rather than $M=0$. Similarly, in the initial gauge approximation, we will include M as part of the vertex R, since it is not determined by the gauge identity. Actually for infrared behaviour magnetic terms are innocuous as they are visibly soft, but for ultraviolet characteristics they become important.

Make the initial substitution (11) in (A.1) to obtain the integral equations for the spectral functions. Dropping the e^4, 2γ-piece, (in analogy to scalar electrodynamics) the problem reduces to

$$Z^{-1}\eta_{\mu\nu} = (\eta_{\mu\nu}p^2 - p_\mu p_\nu)\int ds\,\frac{\rho(s)+\tau(s)}{p^2-s} + m_0^2\int\frac{(-\eta_{\mu\nu}+p_\mu p_\nu/s)\rho(s) - \eta_{\mu\nu}\tau(s)}{p^2-s}\,ds$$

$$+\int\frac{ds}{p^2-s}\,[\rho(s)(-\delta^\lambda_\mu + p_\mu p^\lambda/s)\Pi^R_{\lambda\nu}(p,s) + \tau(s)\Pi^T_{\mu\nu}(p,s)] \quad \text{(A.3)}$$

where Π^R and Π^T are self-energies in lowest-order perturbation theory for a meson of mass $\sqrt{s}$ possessing interactions (A.2) and (10) respectively at one vertex. If we decompose Π into longitudinal and transverse parts,

$$\Pi_{\mu\nu}(p,s) = (-\eta_{\mu\nu} + p_\mu p_\nu/p^2)\Pi_\perp(p^2,s) + (p_\mu p_\nu/p^2)\Pi_\parallel(p^2,s)$$

and take imaginary parts of (A.3), we remain with the coupled scalar equations†

$$(s-m^2)(\rho(s)+\tau(s)) = \frac{1}{\pi}\int_{m^2}^{s}\frac{ds}{s-s'}\,[\rho(s')\,\mathrm{Im}\,\Pi^R_\perp(s,s') - \tau(s')\,\mathrm{Im}\,\Pi^T_\perp(s,s')] \quad \text{(A.4)}$$

$$-m^2\tau(s) = \frac{1}{\pi}\int_{m^2}^{s}\frac{ds'}{s-s'}\left[\left(\frac{s}{s'}-1\right)\rho(s)\,\mathrm{Im}\,\Pi^R_\parallel(s,s') + \tau(s')\,\mathrm{Im}\,\Pi^T_\parallel(s,s')\right].$$

The infrared behaviour concerns the limit $s\to m^2$, whereas when $s\to\infty$ we anticipate that the ultraviolet behaviour will show symptoms of non-renormalisability if developed perturbatively in powers of e^2, starting with $\rho(s)=\delta(s-m^2)$, $\tau(s)=0$. However, let us look for *non-perturbative* solutions of (A.4). The absorptive parts entering there are straightforwardly calculated to equal ($p^2 \geq m^2$),

$$\mathrm{Im}\,\Pi^R_\perp(p^2,m^2) = -\frac{e^2(p^4-m^4)}{96\pi m^2p^4}[12m^2p^2 - (M^2+3M+1)(p^2-m^2)^2 + \tfrac{3}{2}(1-a)(p^2+m^2)^2]$$

† Note the sum rules $Z^{-1} = \int(\rho(s)+\tau(s))\,ds = m_0^2\int s^{-1}\rho(s)\,ds$ which follow from (18). At the level of (A.4) subtractions are made to interpret

$$\Pi(p^2,s) = \frac{p^2-s}{\pi}\int\frac{\mathrm{Im}\,\Pi(s',s)\,ds}{(s'-s)(s'-p^2)}$$

as the renormalised self-energy.

$$\mathrm{Im}\,\Pi_{\parallel}^{R}(p^2, m^2) = \frac{e^2(p^2-m^2)}{32\pi p^4}\{6m^2p^2 + M(p^2-m^2)[(M+1)(p^2-m^2) - 3(p^2+m^2)] - \tfrac{3}{2}(1-a)m^2(p^2+m^2)\} \tag{A.5}$$

$$\mathrm{Im}\,\Pi_{\perp}^{T}(p^2, m^2) = \frac{e^2(p^2-m^2)}{64\pi p^4}[8p^2(p^2+m^2) + \tfrac{1}{3}(p^2-m^2)^2 + (1-a)(p^2+m^2)(3p^2+m^2)]$$

$$\mathrm{Im}\,\Pi_{\parallel}^{T}(p^2, m^2) = \frac{e^2(p^2-m^2)}{64\pi p^4}[(p^2-m^2)^2 - 3(1-a)(p^4-m^4)].$$

Notice the threshold behaviour of $\mathrm{Im}\,\Pi_{\parallel}^{T}$ which, associated with $(p^2/m^2-1)\,\mathrm{Im}\,\Pi_{\parallel}^{R}$, means that τ vanishes relative to ρ as the mass shell is approached. Also, as expected, magnetic M-contributions disappear as $p^2 \to m^2$. In fact, in the infrared limit, the relevant equation simplifies to ($a \neq 3$),

$$(p^2-m^2)\rho(p^2) \simeq \frac{1}{\pi}\int_{m^2}^{p^2} \mathrm{d}s\,\rho(s)\frac{\mathrm{Im}\,\Pi_{\perp}^{R}(p^2, s)}{p^2-s} \underset{p^2\to s}{\simeq} \frac{e^2(a-3)}{8\pi^2}\int_{m^2}^{p^2} \mathrm{d}s\,\rho(s)$$

possessing the solution

$$\rho(p^2) \propto (p^2-m^2)^{-1+e^2(a-3)/8\pi^2} \tag{A.6}$$

and leading to

$$\Delta_{\mu\nu}(p) \underset{p^2\to m^2}{\to} (-\eta_{\mu\nu} + p_\mu p_\nu/m^2)/(p^2-m^2)^{1-e^2(a-3)/8\pi^2}. \tag{A.7}$$

The answer is identical to scalar and spinor electrodynamics, and we conjecture that such behaviour is valid for any spin field.

The ultraviolet limit is not so trivial since we must deal with the coupled pair (A.4). Even in the Fermi gauge ($a = 1$) with M set equal to zero, the reduced equations look pretty formidable:

$$(z-1)(\rho(z)+\tau(z)) = -\frac{e^2}{16\pi^2}\int_1^z \mathrm{d}z'\left\{\rho(z')\left(5-\frac{z'}{z}\right) + \tau(z')\left[2+\frac{2z'}{z}+\frac{1}{3}\left(1-\frac{z'}{z}\right)^2\right]\right\}$$

$$\tau(z) = -\frac{e^2}{16\pi^2}\int_1^z \mathrm{d}z'\left[\rho(z')\left(3-\frac{3z'}{z}\right) + \tau(z')\frac{1}{4}\left(1-\frac{z'}{z}\right)^2\right]$$

since the solutions cannot be simple power dependences $\rho \sim z^r$, $\tau \sim z^t$ even asymptotically.

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